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Comment on "Resolution of Runge-Kutta-Nystrom Condition Equations through Eighth Order"

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AT the time of publication of the subject paper,[1] the author was obliged to resort to a numerical experiment to eliminate two of the condition equations in deriving the general eighth-order, eight-stage algorithms. Since then the author has succeeded in developing an analytic argument which is, of course, far more satisfying and is now presented in this Comment.

As in the paper, we impose the constraints

$$b_1=0, \quad p_7=1$$

Then we form a set of five equations by adding Eqs. (α^i) and (α^{i+2}) and subtracting two times Eq. (α^{i+1}) for $i=1,\ldots,5$. When these are compared to Eqs. $(\gamma^0),(\epsilon^0),(\iota^0),(\sigma^0),(e_2)$, and we recall that $H_1^0=0$, it follows that

$$H_j^0=\tfrac{1}{2}\,p_j(1-p_j)^2 b_j \quad (j=2,\ldots,6)$$

Equation (e_5) can now be written as

$$(1-p_2)^2 A_2^4 b_2+\ldots+(1-p_6)^2 A_6^4 b_6=2/8! \qquad (e_5)$$

which is identical to the equation formed by adding Eqs. (ι^0) and (ι^2) and subtracting two times Eq. (ι^1). Therefore, Eq. (e_5) may be discarded if the three equations (ι) are satisfied by the Nystrom parameters. The same operations applied to Eqs. (λ^0), (λ^1), and (λ^2) show that Eq. (e_6) may also be discarded.

All minimal-stage Runge-Kutta-Nystrom algorithms through eighth order are now complete with the condition equations fully resolved analytically.

Finally, we note a typographical error in the paper on page 1016. Two sets of parameters p_0, p_1, p_2, p_3 were cited which Nystrom used to develop fifth-order, four-stage algorithms. The first set should have been

$$p_0=0, \quad p_1=\frac{2}{5}, \quad p_2=\frac{2}{3}, \quad p_3=\frac{4}{5}$$

The value for p_3 was omitted in the typesetting.

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Comment on "Stress Concentration in the Plastic Range"

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THE authors' rationale for using a semigraphical method for determining the plastic stress concentration factor, k_p, from the Neuber equation used with the Ramberg-Osgood stress-strain law, is the claim that the solution of the resulting equation requires a lengthy trial and error procedure.[1] One purpose of this Comment is to show that the procedure need not be "lengthy" when the equation is expressed in nondimensional form. The time required for hand computation with a desk or pocket electronic calculator is directly related to the desired precision of the solution value. A simple root-finder routine has been used in our laboratory for machine computation as well. Another purpose of this Comment is to identify a limitation in calculating K_p which is a consequence of the limitation on the applicability of the Ramberg-Osgood equation and to refer to some earlier experimental results which were compared with the predictions of the Neuber theory.

Using the same notation as Ref. 1, the Neuber equation can be written as

$$K_{el}^2=\sigma\epsilon/p\epsilon_p \qquad (1)$$

It is assumed that the reference stress, p, is a known quantity. By suitable algebraic manipulation, the Ramberg-Osgood approximation of the stress strain curve can be transformed to

$$\sigma\epsilon=(\sigma_y^2/E)\,[(\sigma/\sigma_y)^2+(3/7)(\sigma/\sigma_y)^{m+1}] \qquad (2)$$

and similarly

$$p\epsilon_p=(\sigma_y{}^2/E)\,[(p/\sigma_y)^2+(3/7)(p/\sigma_y)^{m+1}] \qquad (3)$$

Substitution of Eqs. (2) and (3) into Eq. (1) results in an equation of the form

$$Z^2+(3/7)Z^{m+1}-B=0 \qquad (4)$$

Where $Z=(\sigma/\sigma_y)$ and

$$B=K_{el}^2[(p/\sigma_y)^2+(3/7)(p/\sigma_y)^{m+1}] \qquad (5)$$

The second term in the brackets in Eq. (5) is vanishingly small when p is elastic.

Numerical solutions, accurate to one part in a thousand, can be found in times comparable to that required for the procedure of Ref. 1 on a pocket electronic calculator. For machine computation, I have used a simple root-finder routine as part of a more extensive program for computation

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Table 1 Plastic notch parameters for a titanium alloy structural element with $K_{el}=2$

	Eq. (4)			Eq. (9)			Discrepancy, %	
p/σ_y	σ/σ_y	K_p	K_ϵ	σ/σ_y	K_p	K_ϵ	K_p	K_ϵ
0.598	1.00	1.67	2.39	1.00	1.67	2.39	0	0
.802	1.06	1.33	3.02	1.10	1.37	2.93	2.9	3.1
.889	1.08	1.22	3.28	1.14	1.28	3.13	4.7	4.8
1.000	1.12	1.12	3.56	1.24	1.24	3.24	9.7	9.9
1.100	1.19	1.08	3.69	1.36	1.24	3.24	12.9	13.9

of the various notch parameters as p/σ_y is incremented, simulating a monotonic loading test of a notched tensile specimen.

There is a limitation, however, on the applicability of Eq. (4) based upon the limitations of the Ramberg-Osgood equation in describing stress-strain properties of materials. Analysis of many sets of stress-strain data for steel, titanium, and aluminum have revealed that increasingly larger discrepancies between test data and calculated values occur for data points beyond the yield stress. Post-yield data require a different analytic formulation.[2] Hence, Eq. (4) should not be used for values of (σ/σ_y) greater than unity. If this criterion is applied to the results shown in Figure 2 of Ref. 1, then the following values of (p/σ_y) represent the upper limit of applicability of the analysis:

M value =	3	5	9	199
(p/σ_y) limit =	0.561	0.583	0.596	0.598

The semigraphical K_p results given in the figure for p/σ_y values in excess of those given above must be considered questionable.

It has previously been shown[2] that stress-strain data beyond the yield can be approximated by

$$A=(0.7E/\sigma_y)^n\sigma_y \tag{7}$$

After substitution of Eq. (7) into Eq. (6) and algebraic manipulation of the resulting equation, an analog to Eq. (2) is derived in the form

$$\sigma\epsilon=(\sigma_y^2/E)\,(10/7)\,(\sigma/\sigma_y)^a \tag{8}$$

where $a=(1+n)/n$.

For values of p/σ_y) less than unity, the equation for σ/σ_y is obtained by substituting Eqs. (8) and (3) in Eq. (1) which after rearrangement becomes

$$\sigma/\sigma_y=(.7K_{el}^2[\,(p/\sigma_y)^2+(3/7)\,(p/\sigma_y)^{m+1}\,])^{1/a} \tag{9}$$

For values of p/σ_y equal to or greater than unity, the values of σ/σ_y is given by

$$\sigma/\sigma_y=K_{el}^{2/a}\,(p/\sigma_y) \tag{10}$$

Note that Eq. (10) indicates that K_p and K_ϵ reach limiting values when the reference stress reaches the secant yield stress. The limiting values are given by

$$K_p=K_{el}^{2n/(1+n)} \tag{11a}$$

$$K_\epsilon=K_{el}^{1/(1+n)} \tag{11b}$$

An analysis of tension stress-strain data from titanium alloy TI-8-8-8-2-3 was made using the methods of Ref. 2. The exponents found were $m=19.16$ and $n=0.18090$.

The stress and strain parameters for $K_{el}=2$ were calculated for incremental p/σ_y values using Eq. (4) and Eq. (9) with the initial p/σ_y chosen to yield $\sigma/\sigma_y=1$. The comparison is shown in Table 1. The values calculated from Eq. (4) are the same as those which would be obtained from the semigraphical method of Ref. 1.

The discrepancies between the results is a quantitative indication of the limitation of the Ramberg-Osgood equation in describing the stress-strain properties beyond the yield of the particular titanium alloy chosen. Other materials, particularly those with low m values exhibit the same characteristics.

Two final comments. The authors caution that the semigraphical method may be increasingly more inaccurate for large K_{el}. The direct algebraic method using Eq. (4) or Eq. (9) has no such limitation.

In response to the conclusion of Ref. 1 regarding experimental verification of the Neuber equation, I would refer the authors to an experimental study of notch strain.[3] Comparison between theoretical results, developed using the Neuber equation with a piecewise analytical approximation of stress-strain properties, and experiments indicated that for K_{el} of 1.5 and 2.0, the theory was conservative. Subsequent unpublished results for larger K_{el} up to 6.0 showed the same trend. Additionally, the experiments showed that for any K_{el} value the discrepancy between theory and experiment increased as the load level was increased. The discrepancy may, in part, be the result of using uniaxial stress-strain properties in developing the theoretical results for a stress field in the test specimens which is essentially biaxial.

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Comment on "A Mathematical Description of Gas-Surface Interactions Based on Reciprocity"

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KINSLOW[1] uses the reciprocity principle to develop an interesting model for a scattering kernel to describe reflected velocity distributions in gas-surface interactions.

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